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# THE STABILITY OF TRIANGULAR LIBRATION POINTS IN THE PHOTOGRAVITATIONAL THREE-BODY PROBLEM†

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The photogravitational restricted three-body problem in which a passively gravitating point, in addition to the Newtonian force of gravitation the main bodies, also experiences the force of light pressure from each of them is considered. This problem provides a fairly adequate model, for example, of the motion of a particle of a gas-dust cloud which is in the field of two gravitating and radiating stars. In the case of the elliptic restricted problem when the main bodies are rotating about one another in elliptic orbits, the existence of a family of positions of relative equilibrium is established which are analogous to the Lagrangian libration points of the classical restricted three-body problem. The necessary conditions for the orbital stability of the triangular libration points which have been found are derived using the linearized equations of motion. It is shown that, in the configuration space of the system, the stability domain has a fairly simple geometrical meaning in the circular version of the problem. Conditions for the existence of parametric resonance, which leads to instability in the elliptic version of the problem, are established for small values of the eccentricity. © 2001 Elsevier Science Ltd. All rights reserved.

The photogravitational restricted three-body problem (see, the review [1]) models fairly adequately, for example, the motion of a particle of a gas-dust cloud which is in the field of two gravitating and radiating stars. In this problem, an investigation of the positions of relative equilibrium, in a system of coordinates rotating together with the stars, which are analogous to the collinear and triangular libration points of the classical restricted three-body problem, is of great interest. A fairly full investigation of the family of such points has been carried out in the circular version of the photogravitational problem (including an analysis of their stability in the first approximation) [2–5] and a non-linear analysis has also been given in certain cases of internal resonance.‡ It is of interest to examine the effect of the eccentricity of the orbit of the main bodies on the conditions of the existence and stability of the above-mentioned positions of relative equilibrium. Certain aspects of this problem have been considered previously [6–9]. However, the results are only partial (it was assumed [6] that only one body radiates and in the case of small eccentricities the only conclusion drawn [7] concerned a small change in the stability domain of the circular problem without discovering new instability zones due to the presence of eccentricity and the occurrence of parametric resonance) or they do not give the complete pattern of the stability domain since they show computer calculations of individual points of this domain for certain fixed numerical values of the four parameters of the problem (the two reduction coefficients, the mass parameter and the eccentricity) [8, 9].

The approach proposed here (changing to configuration space) enables one to obtain a fairly simple and physically clear picture of the effect of the eccentricity of the orbit of the main bodies on the position and stability of the triangular libration points when its values are small. New instability zones are revealed in this case, which arise in the case of the eccentricity values which may be as small as desired but do not occur in the circular problem. It is established that this instability is due to the appearance of parametric resonance (it is shown that only one type of such resonance is possible), to which a simple geometric interpretation is also given.

## 1. THE EQUATIONS OF MOTION AND POSITIONS OF RELATIVE EQUILIBRIUM

We shall assume that a repulsive force of light pressure, defined as [2]

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‡TURESHBAYEV, A. T., The stability of the steady-state solutions of the photogravitational restricted three-body problem, Candidate dissertation, 01.02.01, Moscow, 1986.

$$F_p = (1+\varepsilon)sC/R^2 \tag{1.1}$$

where $C$ is a coefficient, characterizing the power of the radiation source, $s$ is the characteristic cross-section area of the particle and $\varepsilon$ is the reflection coefficient ($0 \leqslant \varepsilon \leqslant 1$), acts on a particle $P$ of negligibly small mass $m$ as viewed from a body $S$ of mass $M$, in addition to the gravitational force $F_g = fMm/R^2$ ($R$ is the distance between the bodies).

The total force $F$, acting on the particle $P$, can be represented in the form $F = QF_g$, where $Q$ is a constant coefficient, characterizing the effect of the force of the light pressure, equal to

$$Q = (F_g - F_p)/F_g, \quad -\infty < Q \leqslant 1$$

which is called reduction coefficient of the mass of the particle $P$.

Taking account of (1.1), we obtain

$$Q = 1 - q\sigma, \quad q = C/(fM), \quad \sigma = (1+\varepsilon)s/m \tag{1.2}$$

($\sigma$ is the sail capacity of the particle). As we see, the reduction coefficient $Q$ is not solely determined by the characteristics of the radiation source but also by the sail capacity of the particle $P$. In the case of homogeneous particles of spherical shape with a radius $\rho$ and a density $\delta$, we will have

$$Q = 1 - 3q(1+\varepsilon)/(4\delta\rho)$$

hat is, the effect of the light pressure increases when the absolute dimensions of the particle decrease and can be as large as desired even in the case of particles of high density.

To describe the motion of a passive gravitating point $P$ in a gravitational – repulsive field of two radiating masses $S_1$ and $S_2$, which are rotating around one another in an elliptic orbit with eccentricity $e$, in a Cartesian, barycentric system of coordinates $OXYZ$ which is rotating around the $OZ$ axis (which is used in the classical restricted three-body problem [10]), we will have

$$\ddot{X} - 2\dot{\upsilon}\dot{Y} - \ddot{\upsilon}X = \frac{\partial U}{\partial X}, \quad \ddot{Y} + 2\dot{\upsilon}\dot{X} - \ddot{\upsilon}Y = \frac{\partial U}{\partial Y}, \quad \ddot{Z} = \frac{\partial U}{\partial Z} \tag{1.3}$$

$$U = Q_1\frac{1-\mu}{R_1} + Q_2\frac{\mu}{R_2}, \quad R_j = (X - X_j)^2 + Y^2 + Z^2, \quad j = 1,2$$

Here, $X_1 = -\mu$ and $X_2 = 1 - \mu$ are the dimensionless coordinates of the points $S_1$ and $S_2$ (the sum of their masses and the distance between them $r = p/(1 + e\cos v)$, where $p$, $e$ and $v$ are, respectively, a local parameter, the eccentricity and the true anomaly of their orbit, are adopted as the units of mass and length) and $Q_1$ and $Q_2$($-\infty < Q_j \leqslant 1$) are the reduction coefficients of the mass of the particle $P$ (when $Q_1 = Q_2 = 1$, Eqs (1.3) become the equation of the classical restricted three-body problem). Differentiation with respect to time is indicated by a dot.

In system (1.3), we now change to the new variables $x, y, z$ using the formulae

$$X = rx, \quad Y = ry, \quad Z = rz$$

and, as the new independent variable, we take the true anomaly $v$ (again, we shall denote differentiation with respect to $v$ by a dot). Then, system (1.3) is written as

$$\ddot{x} - 2\dot{y} = f(\upsilon)\frac{\partial W}{\partial x}, \quad \ddot{y} + 2\dot{x} = f(\upsilon)\frac{\partial W}{\partial y}, \quad \ddot{z} = f(\upsilon)\frac{\partial W}{\partial z} \tag{1.4}$$

$$W = \frac{1}{2}(x^2 + y^2 - ez^2\cos\upsilon) + U, \quad f(\upsilon) = (1 + e\cos\upsilon)^{-1}$$

We now consider the positions of relative equilibrium (the libration points) of the particle $P$ in the orbital plane of motion of the main bodies when $e \neq 0$, that is, in the case of an photogravitational elliptic restricted three-body problem (in the case of the circular problem this question is considered in [3–5]). When $z = 0, y = 0$, the coordinates of these points $x_*$ and $y_*$ are found from the system of equations

$$x - Q_1(1-\mu)(x+\mu)/R_1^3 - Q_2\mu(x-1+\mu)/R_2^3 = 0$$
$$1 - Q_1(1-\mu)/R_1^3 - Q_2\mu/R_2^3 = 0 \tag{1.5}$$

containing the three parameters ($Q_1$, $Q_2$, $\mu$). This system is difficult to solve analytically in these coordinates. However, when account is taken of the fact that Eq. (1.5) is linear in the parameters $Q_1$ and $Q_2$, it is easy to find their dependence on the coordinates of the libration points. We will have

$$Q_j = R_j^3, \quad j = 1, 2, \quad R_1 + R_2 \geqslant 1 \tag{1.6}$$

We conclude from this that, for every $\mu$ and any pair $Q_1 > 0$ and $Q_2 > 0$ (that is, only when gravitation predominates over repulsion) for which $Q_1^{1/3} + Q_2^{1/3} \geqslant 1$, a pair of values $R_1$ and $R_2$ is found which determine the two triangular libration points which are symmetrical about the $Ox$ axis. The complete set of such points (which corresponds to different values of $Q_1$ and $Q_2$) entirely fills a domain of the $xy$ plane that is bounded by two circles of unit radius with their centres at the points $S_1$ and $S_2$. Note that, like in the classical case, these points exist both in the circular as well as in the elliptic problem for any eccentricity value.

Since, as follows from relations (1.2), the reduction coefficients are not solely determined by the parameters of the radiation source but also by the sail capacity of the particle $P$, it can be assumed that, for any pair of main bodies $S_1$ and $S_2$ in the indicated domain of existence of triangular libration points, a non-denumerable set of different particles will be found which have a sail capacity corresponding to the reduction coefficient from (1.6).

It follows from relations (1.2) that, for a given pair of bodies $S_1$ and $S_2$, the reduction coefficients of all the particles located at the triangular libration points must satisfy the same relation

$$(1 - Q_1)/(1 - Q_2) = q_1/q_2 = k$$

n which, without loss in generality, it can obviously be assumed that $0 \leqslant k \leqslant 1$. From this, in accordance with (1.6), we obtain the conditions for finding the coordinates of all of the triangular libration points which are possible for the given pair of main bodies characterized by the value of the constant $k$

$$(1 - R_1^3)/(1 - R_2^3) = k \tag{1.7}$$

The curves corresponding to Eq. (1.7) are shown in Fig. 1 for different values of $k$.

## 2. THE STABILITY OF THE TRIANGULAR LIBRATION POINTS IN THE CIRCULAR PROBLEM

We now consider the question of the Lyapunov stability of the triangular libration points for zero and fairly small values of the eccentricity $e$. On introducing the perturbations $x_1 = x - x_*$, $x_2 = y - y_*$, $x_3 = z - z_*$, we obtain the following equations of the perturbed motion in the first approximation (summation is carried out from $j = 1$ to $j = 2$)

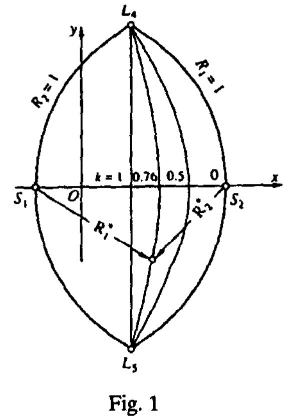


Fig. 1

$$\begin{aligned}
&\ddot{x}_1 - 2\dot{x}_2 + c_{xx}x_1 + c_{xy}x_2 = 0\\
&\ddot{x}_2 + 2\dot{x}_1 + c_{xy}x_1 + c_{yy}x_2 = 0\\
&\ddot{x}_3 + c_{zz}x_3 = 0
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
&c_{xx} = f(\upsilon)\{-1 + \Sigma\Phi_j[R_j^2 - 3(x_* - x_j)^2]\}\\
&c_{xy} = f(\upsilon)[-3y_*\Sigma\Phi_j(R_j - x_j)]\\
&c_{yy} = f(\upsilon)[-1 + \Sigma\Phi_j(R_j^2 - 3y_*^2)], \quad c_{zz} = 1\\
&\Phi_1 = Q_1(1-\mu)/R_1^5, \quad \Phi_2 = Q_2\mu/R_2^5
\end{aligned} \tag{2.2}$$

It is obvious that the coefficients of Eqs (2.1) are analytic functions of the eccentricity $e$ which we shall consider as the small parameter of system (2.1). Then, we know [11] that the characteristics exponents of the periodic system (2.1) when $e \neq 0$ must become the roots of the characteristic equation of the corresponding autonomous system (which is obtained from (2.1) when $e = 0$) which decomposes into a quadratic equation

$$\lambda^2 + 1 = 0$$

and a biquadratic equation

$$\lambda^4 + (4 + c_{xx} + c_{yy})\lambda^2 + (c_{xx}c_{yy} - c_{xy}^2) = 0 \tag{2.3}$$

On eliminating the quantities $Q_1$ and $Q_2$ in the coefficients of these equations using relations (1.6) and putting

$$y_*/R_j = \sin\psi_j, \quad (x_* - x_j)/R_j = (-1)^{j+1}\cos\psi_j, \quad j = 1, 2$$

we obtain expressions for the roots of Eq (2.3) in $\lambda^2$.

$$\lambda_{1,2}^2 = \frac{1}{2}\{-1 \pm [1 - 36\mu(1-\mu)\sin^2(\psi_1 + \psi_2)]^{1/2}\} \tag{2.4}$$

From the condition that the right-hand sides of Eqs (2.4) are negative, we find the necessary conditions for the stability of the triangular libration points when $e = 0$ (the roots of the quadratic equation corresponding to a change in the $z$ coordinate, as in the classical problem, are found to be equal to $\pm i$)

$$36\mu(1-\mu)\sin^2(\psi_1 + \psi_2) \leqslant 1 \tag{2.5}$$

When there is no light pressure, the condition obtained becomes the well-known [10] necessary condition for the stability of the triangular libration points $L_4$ and $L_5$ in the classical restricted three-body problem (for this, it is necessary to put $\psi_1 = \psi_2 = \pi/3$ in (2.5)). For sufficiently small values of $\mu$, determined by the condition

$$36\mu(1-\mu) \leqslant 1$$

all the libration points will be stable. The maximum value $\mu = 1/2 - \sqrt{2}/3 = 0.028595\ldots$, determined by this condition, is somewhat smaller than the corresponding limiting value for stability in the classical problem. However, stability of the libration points is also possible in the problem under consideration for larger values of $\mu$.

In fact, for this it is only necessary that the coordinates of the libration points should satisfy inequality (2.5), which is possible for any $\mu$ for certain $\psi_1$ and $\psi_2$.

It can be shown that, in this case, the stability domain will consist of two parts (they are shown hatched in Fig. 2) which are separated by an instability gap, the boundaries of which are arcs of two circles of the same radius $R = 3\sqrt{\mu(1-\mu)}$ and having the section $S_1S_2$ as a common chord (Fig. 2). This splitting

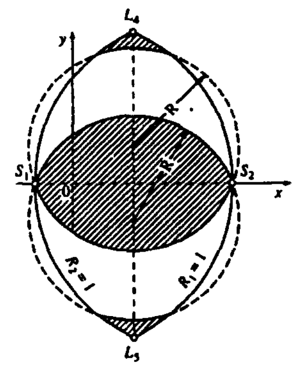

Fig. 2

of the stability domain occurs at values of $\mu$ exceeding the smaller root of the equation (it is clear that it can be assumed that $\mu \leqslant 0.5$ without any loss of generality)

$$36\mu(1-\mu)=1$$

When $\mu$ is increased further the instability gap expands and the part of the stability domain adjacent to the classical libration point becomes smaller, contracting to this point at the other bifurcation value $\mu$ which is the smaller of the roots of the equation

$$27\mu(1-\mu)=1$$

and is identical with the value of $\mu$ which is critical for stability in the classical restricted three-body problem. The part of the stability domain adjacent to the $Ox$ axis, while becoming somewhat smaller in size, remains up to $\mu = 0.5$.

## 3. THE EFFECT OF THE ECCENTRICITY

We will now consider the question of the stability of the triangular libration points in the case of non-zero but fairly small values of the eccentricity $e$ of the orbits of the main bodies. Although the distances between the bodies in the initial system of coordinates $OXYZ$ cannot now remain unchanged, the quantities $R_1$ and $R_2$ will be constant, as before if, at a certain instant of time, they were determined from relations (1.6).

The equations of the perturbed motion will not now be autonomous but their periodic coefficients will contain $2\pi$-periodic functions of the true anomaly $\nu$ which vanish when $e = 0$ and for fairly small eccentricity values which are being analytic functions of the latter. It is known [11] that the characteristic exponents of the class of Hamiltonian system under consideration, when there is no parametric resonance, differ from the roots of the characteristic equation of the corresponding autonomous system (which is obtained from the periodic system when the small parameter vanishes) by an amount of the order of magnitude of the square of the small parameter, the role of which is played by the eccentricity in the problem being considered. It can be concluded from this that the stability domain of the triangular libration points for small values of $e$ will differ only slightly from their stability domain constructed above for the case when $e = 0$. However, we know [11, 12] that, even for values of the small parameter as small as desired, zones of instability can arise in the above mentioned domain which correspond to cases of parametric resonance when the characteristic exponents of the system when $e = 0$ satisfy one of the conditions

$$2\omega_\alpha = p,\quad \omega_\alpha + \omega_\beta = p,\quad p = 1,2,3,\ldots \tag{3.1}$$

When $e = 0$, these characteristic exponents become the roots of the characteristic equation (2.3) which, according to (2.4), will be

$$\omega_\alpha = (-1)^\alpha\,|\lambda_\alpha|,\quad \alpha = 1,2$$

Hence, the zone of parametric resonance instability when $e \neq 0$ will be generated from the parametric resonance curves (3.1) when $e = 0$. From elementary analysis it can be shown that, of all the possible cases of parametric resonance (3.1), as in the classical restricted three-body problem [12], only one case

$$4\omega_2^2 \equiv 2\{1 - [1 - 36\mu(1-\mu)\sin^2(\psi_1 + \psi_2)]^{1/2}\} = 1 \tag{3.2}$$

is realized when $e = 0$.

It follows from the theory of the parametric resonance of Hamiltonian systems [11, 12] (and the system being considered can be represented in Hamiltonian form) that, in order for instability at small $e$ actually to occur and for relation (3.2) to be satisfied, it is necessary that the harmonics $\cos\nu$, $\sin\nu$ should certainly appear in the expansion of the periodic coefficients of system (2.1) in a Fourier series which, obviously, will also hold, as can be seen from expression (2.2) for these coefficients.

We conclude from this that the whole set of libration points which are unstable in the first approximation in the case of small but non-zero values of the eccentricity will lie in a region, adjacent to the curve defined by Eq. (3.2) which reduces to the form

$$48\mu(1-\mu)\sin^2(\psi_1 + \psi_2) = 1 \tag{3.3}$$

It can be seen from (3.3) that this instability only arises for values of $\mu$ from the range $\mu^* \leqslant \mu \leqslant 0.5$ where $\mu^* = 1/2 - \sqrt{11/3}/4 = 0.0212865\ldots$ is the smaller root of the equation

$$48\mu(1-\mu) = 1$$

The resulting equation, like the equation of the boundary of the stability domain (2.5), describes two arcs of a circles of the same radii $R = 2\sqrt{3\mu(1-\mu)}$ with a common chord which is the motion $S_1S_2$, and, close to these arcs for sufficiently small values of $e \neq 0$, the triangular libration points will be unstable in the Lyapunov sense, and also in the strict sense (that is, by virtue of the complete equations of the perturbed motion) as a consequence of the occurrence of parametric resonance. It can be seen that such a case of instability will hold for any pair of bodies $S_1$ and $S_2$ and, at least, for those libration points coordinates of which satisfy condition (1.7) and equality (3.3) simultaneously.

Note that the instability of triangular libration points which has been revealed is a generalization of the case of instability caused by precisely the same type of parametric resonance that also occurs in the classical elliptic restricted three-body problem which is obtained from the problem under consideration when $Q_1 = Q_2 = 1$ [12, 13]. It is interesting that the corresponding resonance value of $\mu$ is identical to its bifurcation value at which a splitting of the stability domain occurs in the circular photogravitational problem. We further note that this interpretation of the stability domain and of the case of instability caused by parametric resonance enables one to obtain, in a very simple manner, the results in [6], where radiation from just one of the main bodies was taken into account (that is, one of the reduction coefficients is equal to unity). In this case, all the triangular libration points are arranged on one of the circles of unit radius. They are all stable when $\mu \leqslant 1/2 - \sqrt{2}/3$ while, for large values, only those of them which are located on the part of the above-mentioned circle which is adjacent to the points $L_4$ and $L_5$ will be stable. Unlike in the general case, parametric resonance occurs here only in the neighbourhood of a single point (the second arc, which corresponds to parametric resonance, does not intersect the above-mentioned circle of unit radius). This interpretation also enables one to carry out a further numerical analysis of the stability domain more purposefully in the case of arbitrary values of the eccentricity which was commenced in [8, 9].

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